Indian Statistical Institute, Bangalore
B. Math II, First Semester, 2022-23
Mid-semester Examination, Introduction to Statistical Inference

21.09.22 **Maximum Score 60** **Duration: 3 Hours**

1. (12+8) Let $X_1, \cdots, X_n$ be iid observations from a beta $(\theta, 1)$ distribution.
   (a) Find the MLE of θ. Use this to find the MLE, T, of $1/\theta$. Is T unbiased for $1/\theta$? Does T attain the Cramer Rao lower bound?
   (b) Find $q(\theta) = E(X_1)$. Does $\bar{X}$ attain the Cramer Rao lower bound as an unbiased estimator for $q(\theta)$?

2. (10) Suppose $X_1, X_2, \cdots, X_n$ are independent random variables such that
$$P(X_i = 1) = p_i = 1 - P(X_i = 0),$$
where $p_1, p_2, \cdots, p_n \in (0, 1)$ are all distinct and unknown. Consider $X = \sum_{i=1}^{n} X_i$ and another random variable Y which is distributed as Binomial(n, $\bar{p}$), where $\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i$. Between X and Y , which is a better estimator of $\sum_{i=1}^{n} p_i$ in terms of their respective mean squared errors?

3. (10) Suppose $\delta(X)$ is a Bayes estimator of parameter $g(\theta)$ under prior π. Suppose $\delta(X)$ is also unbiased for $g(\theta)$.
   (a) Show, by conditioning on X, that $E(\delta(X)g(\theta)) = E(\delta^2(X))$.
   (b) Show, by conditioning on θ, that $E(\delta(X)g(\theta)) = E(g^2(\theta))$.
   (c) Conclude that $E(\delta(X) - g(\theta))^2 = 0$.
   (d) If $X_1, \cdots, X_n$ are iid $\mathcal{N}(\mu, 1)$, then use the above result to conclude that $\bar{X}$ cannot be a Bayes estimator for μ under any prior.
   (e) What can you conclude about unbiased Bayes estimators?

4. (5+10+5)
   (a) Let $f(x;\theta)$ be the pdf or pmf of the sample X and $T(x)$ is a sufficient statistic. Show that for every two sample points x and y, $T(x) = T(y)$ implies that the ratio $f(x;\theta)/f(y;\theta)$ is constant as a function of θ.
   (b) Suppose we have a sufficient statistic $T(x)$ such that for every two sample points x and y, the ratio $f(x;\theta)/f(y;\theta)$ is constant as a function of θ implies $T(x) = T(y)$. Show that $T(x)$ is minimal sufficient.
   (c) Let $X_1, \cdots, X_n$ be iid observations from a normal distribution with mean μ and variance σ^2. Show that $(\bar{x}, s^2)$ is a sufficient statistic for (μ, σ^2). Use the above result to show that it is minimal sufficient.